

# ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER SPHERE

J.-H. ESCHENBURG AND M. KERIN

**ABSTRACT.** Gromoll and Meyer have represented a certain exotic 7-sphere  $\Sigma^7$  as a biquotient of the Lie group  $G = Sp(2)$ . We show for a 2-parameter family of left invariant metrics on  $G$  that the induced metric on  $\Sigma^7$  has strictly positive sectional curvature at all points outside four subvarieties of codimension  $\geq 1$  which we describe explicitly.

## 1. INTRODUCTION

Let  $G = Sp(2)$  be the Lie group of unitary quaternionic  $2 \times 2$ -matrices. Consider the subgroup  $U \subset G \times G$ ,

$$U = \{((\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}), (\begin{smallmatrix} & q \\ & q \end{smallmatrix})) ; q \in Sp(1)\}, \quad (1.1)$$

which acts on  $G$  by left and right translations. D. Gromoll and W. Meyer [5] have shown that this action is free and that the orbit space  $M = G/U$  is a smooth manifold which is an exotic 7-sphere (homeomorphic but not diffeomorphic to the standard 7-sphere). If  $G$  is equipped with a Riemannian metric of nonnegative sectional curvature whose isometry group contains  $U$ , then by O'Neill's formula [1] the orbit space  $M = G/U$  inherits a Riemannian metric of nonnegative sectional curvature. Thus starting with the bi-invariant metric on  $G$ , Gromoll and Meyer constructed a metric of nonnegative sectional curvature on the exotic sphere  $M$ . In fact the curvature is strictly positive on some nonempty open subset of  $M$ . However, as was observed by F. Wilhelm [7], there is also an open subset with zero curvature planes in the tangent space of each of its points. But Wilhelm constructed another  $U$ -invariant metric on  $Sp(2)$  (neither left nor right invariant) for which the curvature of  $M$  is strictly positive outside a subset of measure zero in  $M$  ("almost positive curvature"). In [4] the same fact was claimed for a much simpler and left invariant metric on  $Sp(2)$ ; however, as was pointed out by the second author, the proof contains a serious mistake (see Remark 3 at the end of the present paper). The purpose of our paper is to correct this error. In fact we prove the following result, some ideas of which go back to [3] (see Theorem 4.6 for details):

**Theorem 1.1.** *There is a left invariant and  $U$ -invariant metric on  $G = Sp(2)$  such that the induced metric on  $M = G/U$  has strictly positive curvature outside a finite union of subvarieties of codimension  $\geq 1$ . The zero curvature set  $Z \subset M$  can be explicitly determined.*

---

*Date:* February 2, 2008.

*2000 Mathematics Subject Classification.* 53C20, 53C30.

*Key words and phrases.* Biquotients, Lie groups, left invariant metrics, Quaternions.

The second author would like to thank the University of Pennsylvania for financial support.

## 2. CHEEGER METRICS ON LIE GROUPS

On each Riemannian manifold, let us denote

$$\begin{aligned}\kappa(X, Y) &= \langle R(X, Y)Y, X \rangle, \\ \sec(X, Y) &= \kappa(X, Y)/|X \wedge Y|^2\end{aligned}\tag{2.1}$$

for any two tangent vectors  $X, Y$ ; the second expression is the sectional curvature of the plane  $\sigma$  spanned by  $X, Y$ .

Let  $G$  be a Lie group with a left invariant metric  $\langle \cdot, \cdot \rangle$  of nonnegative sectional curvature. Suppose that the metric is also right invariant with respect to a compact subgroup  $K \subset G$ , hence the induced metric on  $K$  is bi-invariant. The Lie algebras of  $G$  and  $K$  will be denoted  $\mathfrak{g}$  and  $\mathfrak{k}$ . We may contract the metric on  $G$  in the direction of the  $K$ -cosets by viewing  $G$  as the homogeneous space  $(G \times K)/\Delta K$  (where  $\Delta K = \{(k, k); k \in K\}$ ) and choosing the metric induced from the Riemannian product metric on  $G \times sK$  (*Cheeger contraction*, cf. [2], [1]) where  $sK$  is  $K$  with metric scaled by  $s > 0$ . A vector  $(X, X') \in \mathfrak{g} \times \mathfrak{k}$  is perpendicular to the  $\Delta K$ -orbit (“horizontal”) iff  $X + sX' \perp \mathfrak{k}$ , i.e.  $X' = -s^{-1}X_{\mathfrak{k}}$  where  $X_{\mathfrak{k}}$  is the  $\mathfrak{k}$ -projection of  $X$ . Using the Riemannian submersion  $G \times K \rightarrow G$ ,  $(g, k) \mapsto gk^{-1}$ , a horizontal vector  $(X, -s^{-1}X_{\mathfrak{k}}) \in \mathfrak{g} \times \mathfrak{k}$  is mapped onto  $X + s^{-1}X_{\mathfrak{k}} = X_{\perp} + (1 + s^{-1})X_{\mathfrak{k}} \in \mathfrak{g}$  where  $X_{\perp} = X - X_{\mathfrak{k}} \in \mathfrak{k}^{\perp}$ . Vice versa, the horizontal lift of  $X = X_{\perp} + X_{\mathfrak{k}} \in \mathfrak{g}$  is the horizontal vector

$$\begin{aligned}\hat{X} &= (\tilde{X}, -s^{-1}\tilde{X}_{\mathfrak{k}}), \quad \text{where} \\ \tilde{X} &= X_{\perp} + \frac{s}{s+1}X_{\mathfrak{k}}.\end{aligned}\tag{2.2}$$

Thus the new (left invariant) metric is

$$\begin{aligned}\langle X, Y \rangle_1 &= \langle \hat{X}, \hat{Y} \rangle \\ &= \langle \tilde{X}, \tilde{Y} \rangle + s \langle s^{-1}\tilde{X}_{\mathfrak{k}}, s^{-1}\tilde{Y}_{\mathfrak{k}} \rangle \\ &= \langle \tilde{X}, \tilde{Y} \rangle + s^{-1} \langle \tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}} \rangle \\ &= \langle \tilde{X}_{\perp}, \tilde{Y}_{\perp} \rangle + s^{-1}(s+1) \langle \tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}} \rangle \\ &= \langle X_{\perp}, Y_{\perp} \rangle + s(s+1)^{-1} \langle X_{\mathfrak{k}}, Y_{\mathfrak{k}} \rangle \\ &= \langle \tilde{X}, Y \rangle.\end{aligned}\tag{2.3}$$

For the curvature terms we have

$$\kappa(\hat{X}, \hat{Y}) = \kappa(\tilde{X}, \tilde{Y}) + s^{-3} \kappa(\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}).\tag{2.4}$$

Since all terms are nonnegative, the left hand side vanishes if and only if both summands on the right are zero. Thus a plane  $\sigma$  spanned by  $X, Y \in \mathfrak{g}$  has zero curvature in the new metric,  $\sec_1(\sigma) = 0$ , if and only if  $\sec(\tilde{\sigma}) = 0$  and  $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ .<sup>1</sup>

**Example 1.** Suppose that the initial metric  $\langle \cdot, \cdot \rangle$  on  $G$  is bi-invariant. Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be the orthogonal decomposition. Consider the above metric

$$\langle X, Y \rangle_1 = \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle + \tilde{s} \langle X_{\mathfrak{k}}, Y_{\mathfrak{k}} \rangle\tag{2.5}$$

with  $\tilde{s} = \frac{s}{s+1}$ . Then  $\sec(\tilde{\sigma}) = 0 \iff [\tilde{X}, \tilde{Y}] = 0$ , and hence  $\sec_1(\sigma) = 0 \iff$

$$[\tilde{X}, \tilde{Y}] = 0, \quad [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0.$$

---

<sup>1</sup>The “if” statement is not obvious because of the nonnegative O’Neill term. However, in all our examples starting with a bi-invariant metric on some Lie group, the vanishing of the curvature implies that the O’Neill term also vanishes, see [3], p. 29f, Equations (1) - (4) or [8], [6]

If  $(G, K)$  is a symmetric pair, i.e. the orthogonal complement  $\mathfrak{p} \subset \mathfrak{g}$  satisfies  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ , then  $[\tilde{X}, \tilde{Y}]_{\mathfrak{k}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] + [\tilde{X}_{\mathfrak{p}}, \tilde{Y}_{\mathfrak{p}}]$  and  $[\tilde{X}, \tilde{Y}]_{\mathfrak{p}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{p}}] + [\tilde{X}_{\mathfrak{p}}, \tilde{Y}_{\mathfrak{k}}]$ , hence  $\sec_1(\tilde{\sigma}) = 0 \iff$

$$0 = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [X_{\mathfrak{p}}, Y_{\mathfrak{k}}] = [X, Y]. \quad (2.6)$$

**Example 2.** Let  $G \supset K \supset H$  a chain of subgroups and suppose that both  $(G, K)$  and  $(K, H)$  are symmetric pairs. Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  and  $\mathfrak{k} = \mathfrak{h} + \mathfrak{q}$  be the corresponding decompositions. Start with the metric  $\langle \cdot, \cdot \rangle_1$  defined by Example 1, depending on a parameter  $s > 0$ , and define the metric  $\langle \cdot, \cdot \rangle_2$  by Cheeger contraction along  $H$  (depending on a new parameter  $t > 0$ ) as in (2.3) where  $K$  is replaced by  $H$  and  $\langle \cdot, \cdot \rangle_1$  takes the role of  $\langle \cdot, \cdot \rangle$ :

$$\begin{aligned} \langle X, Y \rangle_2 &= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle_1 + \langle X_{\mathfrak{q}}, Y_{\mathfrak{q}} \rangle_1 + \tilde{t} \langle X_{\mathfrak{h}}, Y_{\mathfrak{h}} \rangle_1 \\ &= \langle X_{\mathfrak{p}}, Y_{\mathfrak{p}} \rangle + \tilde{s} \langle X_{\mathfrak{q}}, Y_{\mathfrak{q}} \rangle + \tilde{s} \tilde{t} \langle X_{\mathfrak{h}}, Y_{\mathfrak{h}} \rangle \end{aligned} \quad (2.7)$$

with  $\tilde{t} = \frac{t}{t+1}$ . Then  $\sec_2(\sigma) = 0 \iff \sec_1(\tilde{\sigma}) = 0$  and  $[\tilde{X}_{\mathfrak{h}}, \tilde{Y}_{\mathfrak{h}}] = 0 \iff$

$$0 = [\tilde{X}, \tilde{Y}] = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = [X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}], \quad (2.8)$$

where  $\tilde{X} = X_{\mathfrak{p}} + X_{\mathfrak{q}} + \frac{t}{t+1} X_{\mathfrak{h}}$  and  $\tilde{Y} = Y_{\mathfrak{p}} + Y_{\mathfrak{q}} + \frac{t}{t+1} Y_{\mathfrak{h}}$  like in (2.2).

### 3. ZERO CURVATURE PLANES ON $Sp(2)$

Let us consider the chain  $G \supset K \supset H$  for  $G = Sp(2)$ ,  $K = Sp(1) \times Sp(1)$  and  $H = \Delta Sp(1) = \{ \begin{pmatrix} q & \\ & q \end{pmatrix}; q \in Sp(1) \}$ . The pairs  $(G, K)$  and  $(K, H)$  are symmetric, corresponding to the rank-one symmetric spaces  $S^4$  and  $S^3$ . We start with the bi-invariant trace metric  $\langle X, Y \rangle = \text{Re trace } X^* Y = \text{Re} \sum \overline{x_{ij}} y_{ij}$  on  $\mathfrak{g} = \mathfrak{sp}(2)$ , apply Cheeger contraction in the  $K$ -direction and Cheeger-contract again in the  $H$ -direction, defining metrics  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$  as in Example 2.

Since  $G/K = S^4$  as well as  $K/H = S^3$  and  $H = S^3$  have positive curvature, the vanishing of the last three brackets in (2.8) means the linear dependence of the factors. In particular we may assume  $Y_{\mathfrak{p}} = 0$ , i.e.  $\tilde{Y} = \tilde{Y}_{\mathfrak{k}} = \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix}$ .

**Case 1.**  $X_{\mathfrak{p}} = 0$ , i.e.  $\tilde{X} = \tilde{X}_{\mathfrak{k}} = \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix}$ .

From  $[\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] = 0$  we obtain that the imaginary quaternions  $x_1, y_1$  as well as  $x_2, y_2$  are linearly dependent. Moreover, from  $[X_{\mathfrak{q}}, Y_{\mathfrak{q}}] = [X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$  we see that also  $x_1 \pm x_2$  and  $y_1 \pm y_2$  are linearly dependent. Putting  $y = y_1$ , we may assume

$$\tilde{Y} = \begin{pmatrix} y & \\ & 0 \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & \\ & y \end{pmatrix}. \quad (3.1)$$

**Case 2.**  $X_{\mathfrak{p}} \neq 0$ , i.e.  $X = \begin{pmatrix} x_1 & -\bar{x} \\ x & x_2 \end{pmatrix}$  with  $x \neq 0$ :

Then  $0 = [\tilde{X}, \tilde{Y}]_{\mathfrak{p}} = [X_{\mathfrak{p}}, \tilde{Y}] \iff y_2 = xyx^{-1}$  for  $y := y_1$ , and  $0 = [\tilde{X}, \tilde{Y}]_{\mathfrak{k}} = [\tilde{X}_{\mathfrak{k}}, \tilde{Y}_{\mathfrak{k}}] \iff x_1 = \alpha y_1, x_2 = \beta y_2$  for real numbers  $\alpha, \beta$ , hence

$$\tilde{Y} = \begin{pmatrix} y & \\ & xyx^{-1} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} \alpha y & -\bar{x} \\ x & -\alpha xyx^{-1} \end{pmatrix} \quad (3.2)$$

where  $x, y \in \mathbb{H}$ ,  $y$  imaginary and  $\alpha \in \mathbb{R}$ ; we have  $\beta = -\alpha$  since we require  $\langle \tilde{X}, \tilde{Y} \rangle = 0$ .

**Case 2a.**  $\alpha = 0$ , hence

$$\tilde{Y} = \begin{pmatrix} y & \\ & yxx^{-1} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} & -\bar{x} \\ x & \end{pmatrix}. \quad (3.3)$$

**Case 2b.**  $\alpha \neq 0$ , hence (without loss of generality)  $\alpha = 1$ .

Then  $[X_{\mathfrak{h}}, Y_{\mathfrak{h}}] = 0$  iff  $y + yxx^{-1}$  and  $y - yxx^{-1}$  are proportional which means  $yx^{-1} = \beta y$ . Comparing the norms on both sides we get

$$yx^{-1} = \pm y, \quad (3.4)$$

and

$$\tilde{Y} = Y_{\pm} = \begin{pmatrix} y & \\ & \pm y \end{pmatrix}, \quad \tilde{X} = X_{\pm} = \begin{pmatrix} y & -\bar{x} \\ x & \mp y \end{pmatrix}. \quad (3.5)$$

**Lemma 3.1.** *The zero curvature planes in  $\mathfrak{g} = T_e G$  for  $G = Sp(2)$  and the metric  $\langle \cdot, \cdot \rangle_2$  are spanned by  $X, Y \in \mathfrak{g}$  with  $\tilde{X}, \tilde{Y}$  given by either (3.1) or (3.3) or (3.5).*

#### 4. THE GROMOLL-MEYER SPHERE

The metric  $\langle \cdot, \cdot \rangle_2$  on  $G = Sp(2)$  is invariant under the action of  $U$  (cf. (1.1)) and hence it induces a metric on the orbit space  $M = G/U$ . Consider any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \quad (4.1)$$

Since  $g$  is unitary, the rows and columns are unit vectors, in particular

$$|a|^2 + |b|^2 = 1. \quad (4.2)$$

The vertical space at  $g$  of the submersion  $\pi : G \rightarrow G/U$  is  $T_g(U.g) = gV_g$  with  $V_g = \{v_g; v \in \text{Im } \mathbb{H}\}$  where

$$v_g = g^{-1} \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} g - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \bar{a}va - v & \bar{a}vb \\ \bar{b}va & \bar{b}vb - v \end{pmatrix} \quad (4.3)$$

Thus according to (2.3), a vector  $gX \in T_g G$  is horizontal for  $\pi$  iff

$$0 = \langle X, v_g \rangle_2 = \langle \tilde{X}, v_g \rangle_1 \quad (4.4)$$

for all  $v \in \text{Im } \mathbb{H}$ . Note that  $\langle \tilde{X}, v_g \rangle_1$  is just a multiple of  $\langle \tilde{X}, v_g \rangle$  if one of the components of  $\tilde{X} = \tilde{X}_{\mathfrak{p}} + \tilde{X}_{\mathfrak{k}}$  are zero. Now we discuss which of the zero curvature planes in  $G = Sp(2)$  (see Lemma 3.1) can be horizontal at any  $g \in G$ . By a slight abuse of language, a plane  $\tilde{\sigma}$  spanned by  $\tilde{X}, \tilde{Y} \in \mathfrak{g}$  will be called *horizontal at  $g$*  if

$$\langle \tilde{X}, v_g \rangle_1 = \langle \tilde{Y}, v_g \rangle_1 = 0 \quad (4.5)$$

for all  $v \in \text{Im } \mathbb{H}$ .

**Case 1.**

**Lemma 4.1.** *A plane of type (3.1) is nowhere horizontal.*

*Proof.*  $\langle \tilde{Y}, v_g \rangle = \langle y, \bar{a}va - v \rangle = \langle ay\bar{a} - y, v \rangle$  vanishes for all  $v \in \text{Im } \mathbb{H}$  iff  $y = ay\bar{a}$ , and likewise  $\langle \tilde{X}, v_g \rangle$  vanishes for all  $v$  iff  $y = by\bar{b}$ . But this implies  $|a| = |b| = 1$  in contradiction to (4.2).  $\square$

**Case 2a.**

**Lemma 4.2.** *If a plane of type (3.3) is horizontal at  $g$  then either  $a = 0$  or  $b = 0$  or*

$$\det(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1})) = 0. \quad (4.6)$$

*Proof.* The matrix  $\tilde{X}$  is horizontal at  $g$  if and only if

$$0 = \langle \tilde{X}, v_g \rangle = 2\langle x, \bar{b}va \rangle = 2\langle bx\bar{a}, v \rangle \quad (4.7)$$

for all  $v \in \text{Im } \mathbb{H}$ . This is equivalent to  $bx\bar{a} \in \mathbb{R}$ . Hence, either  $a = 0$  or  $b = 0$  or  $bx = ra$  for some non-zero  $r \in \mathbb{R}$ . In the latter case we have, in particular

$$\text{Ad}(bx) = \text{Ad}(a), \quad (4.8)$$

$$\text{Ad}(x) = \text{Ad}(b^{-1})\text{Ad}(a), \quad (4.9)$$

provided that  $b \neq 0$ . On the other hand, the matrix  $\tilde{Y}$  is horizontal at  $g$  if and only if

$$0 = \langle \tilde{Y}, v_g \rangle = \langle |a|^2 \text{Ad}(a)y - y + |b|^2 \text{Ad}(bx)y - \text{Ad}(x)y, v \rangle \quad (4.10)$$

for all  $v \in \text{Im } \mathbb{H}$ . Since  $y \in \text{Im } \mathbb{H}$ , this means

$$\begin{aligned} 0 &= |a|^2 \text{Ad}(a)y + |b|^2 \text{Ad}(bx)y - y - \text{Ad}(x)y \\ &\stackrel{(4.8)}{=} \text{Ad}(a)y - y - \text{Ad}(x)y \\ &\stackrel{(4.9)}{=} \text{Ad}(a)y - y - \text{Ad}(b^{-1})\text{Ad}(a)y \end{aligned} \quad (4.11)$$

where we have also used  $|a|^2 + |b|^2 = 1$  (4.2). If  $a \neq 0$ , we obtain from the last equality

$$\text{Ad}(a)y \in \ker(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1}))$$

and in particular

$$\det(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1})) = 0. \quad (4.6)$$

□

**Lemma 4.3.** *There exists a plane of type (3.3) which is horizontal at  $g$  if and only if either (4.6) holds or*

$$a = 0, \quad |\text{Im } b| \geq \frac{1}{2} \quad \text{or} \quad b = 0, \quad |\text{Im } a| \geq \frac{1}{2}. \quad (4.12)$$

*Proof.* Suppose first  $a, b \neq 0$ . If (4.6) is satisfied, there is a non-zero  $w \in \ker(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1}))$ . Then defining  $y = \text{Ad}(a^{-1})w$  and  $x = b^{-1}a$ , we obtain a horizontal plane of type (3.3) at  $g$ . The converse conclusion was done before.

Now suppose  $b = 0$ . Then  $|a| = 1$  and Equation (4.11) becomes

$$\text{Ad}(a)y - y = \text{Ad}(x)y. \quad (4.13)$$

Geometrically, this equality means that  $\text{Ad}(a)$  rotates  $y$  by the angle  $\frac{\pi}{3}$  (the three vectors  $\text{Ad}(a)y, y, \text{Ad}(x)y$  form the sides of an equilateral triangle). Hence (4.13) has a solution  $(x, y)$  if and only if the rotation angle of the rotation  $\text{Ad}(a)$  is  $\geq \frac{\pi}{3}$ . This in turn is equivalent to  $\angle(a, 1) \geq \frac{\pi}{6}$ , i.e.  $|\text{Im } a| \geq \frac{1}{2}$ . Inserting the solution  $(x, y)$  into (3.3) defines a horizontal plane of type (3.3). The case  $a = 0$  is similar. □

**Case 2b.**

**Lemma 4.4.** *If a plane of type (3.5) is horizontal at  $g$ , then*

$$|a| = |b| = 1/\sqrt{2} \quad (4.14)$$

and  $w := \operatorname{Im} a^{-1}b$  satisfies

$$\langle w - 2a^{-1}wa, w \rangle = 0. \quad (4.15)$$

*Proof.*

$$\langle v_g, Y_+ \rangle = \langle \bar{a}va + \bar{b}vb - 2v, y \rangle = \langle v, ay\bar{a} + by\bar{b} - 2y \rangle \quad (4.16)$$

$$\langle v_g, Y_- \rangle = \langle \bar{a}va - \bar{b}vb, y \rangle = \langle v, ay\bar{a} - by\bar{b} \rangle \quad (4.17)$$

Thus  $\langle \tilde{Y}, V_g \rangle = 0$  iff one of the following equations holds:

$$\begin{aligned} ay\bar{a} + by\bar{b} &= 2y, \\ ay\bar{a} - by\bar{b} &= 0. \end{aligned}$$

The first of these equations is impossible by the triangle inequality together with (4.2):

$$|ay\bar{a} + by\bar{b}| \leq |ay\bar{a}| + |by\bar{b}| \leq (|a|^2 + |b|^2)|y| = |y| < |2y|.$$

Thus we are left with the second equation,

$$ay\bar{a} = by\bar{b}, \quad (4.18)$$

which implies  $|a| = |b|$ .

Note that we have also shown that  $Y_+$  cannot be horizontal. Thus we need only consider  $\tilde{X} = X_-$  and  $\tilde{Y} = Y_-$  in (3.5), and

$$xyx^{-1} = -y \quad (4.19)$$

which means that  $x$  is imaginary and nonzero with  $x \perp y$ .

Now let  $\tilde{X}, \tilde{Y}$  be as above spanning  $\tilde{\sigma}$ . By the preceding remark we have

$$\tilde{Y} = \begin{pmatrix} y & \\ & -y \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} y & x \\ x & y \end{pmatrix} \quad (4.20)$$

with  $y \perp x \in \operatorname{Im} \mathbb{H}$ . Thus according to (2.5) we get for all  $v \in \operatorname{Im} \mathbb{H}$

$$\begin{aligned} 0 = \langle \tilde{X}, v_g \rangle_1 &= 2\langle x, \bar{b}va \rangle + \tilde{s}\langle y, \bar{a}va + \bar{b}vb - 2v \rangle \\ &= 2\langle bx\bar{a}, v \rangle + \tilde{s}\langle ay\bar{a} + by\bar{b} - 2y, v \rangle \\ &= \langle bxa^{-1} + \tilde{s}(aya^{-1} - 2y), v \rangle, \end{aligned} \quad (4.21)$$

where we have used  $2\bar{a} = a^{-1}$  and  $ay\bar{a} = by\bar{b} = \frac{1}{2}aya^{-1}$  from (4.14) and (4.18). Putting  $p = a^{-1}b/\tilde{s}$ , we obtain

$$\operatorname{Im} apxa^{-1} = 2y - aya^{-1}. \quad (4.22)$$

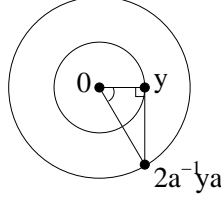
From  $aya^{-1} = byb^{-1}$  we see  $yp = py$ , thus  $p \in \mathbb{C}_y := \mathbb{R} + \mathbb{R}y$  and thus the left multiplication with  $p$  preserves  $\mathbb{C}_y$  and  $\mathbb{C}_y^\perp$ . By (4.19) we have  $x \in \mathbb{C}_y^\perp$  and therefore  $px \in \mathbb{C}_y^\perp$ . Conjugating (4.22) by  $a^{-1}$  we obtain

$$2a^{-1}ya - y = \operatorname{Im}(px) \perp y, \quad (4.23)$$

$$\langle 2a^{-1}ya - y, y \rangle = 0. \quad (4.24)$$

Since  $w = \operatorname{Im} \tilde{s}p \in \mathbb{C}_y$  is a multiple of  $y$ , we may replace  $y$  by  $w$  in Equation (4.24) and obtain (4.15).  $\square$

**Remark 1.**



Geometrically, (4.24) means that the angle between  $y$  and  $a^{-1}ya$  is  $\pi/3 = 60^\circ$ . Thus the rotation angle of  $\text{Ad}(a^{-1})$  (and of  $\text{Ad}(b^{-1})$ , see (4.18)) must be  $\geq \pi/3$ , hence  $\angle(1, a) \geq \pi/6$ , or in other words,

$$\frac{|\text{Im } a|}{|a|} \geq \frac{1}{2}. \quad (4.25)$$

**Lemma 4.5.** *Suppose that  $a, b \in \mathbb{H}$  satisfy (4.14), (4.15) and (4.25). Then there exists a horizontal plane of type (3.5) at  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .*

*Proof.* First suppose that  $\tilde{p} = a^{-1}b = \tilde{s}p$  is real which in view of (4.14) means  $a = \pm b$ . By (4.25), the rotation angle of  $\text{Ad}(a^{-1})$  is  $\geq \pi/3$ , hence there exists a nonzero  $y \in \text{Im } \mathbb{H}$  which is rotated precisely by the angle  $\pi/3$  and thus satisfies (4.24). Put  $x = 2a^{-1}ya - y \perp y$  and define  $\tilde{X}, \tilde{Y}$  as in (4.20). This matrix pair is of type (3.5), and it is perpendicular to  $V_g$  by (4.17) and (4.21).

Now suppose that  $w = \text{Im } \tilde{p} \neq 0$ ; in this case (4.15) implies (4.25). Then we choose  $y = w$  and  $x = \text{Im } (p^{-1}(2a^{-1}wa - w))$ , compare (4.23). Since  $w - 2a^{-1}wa \in \mathbb{C}_y^\perp$  (it is imaginary and perpendicular to  $w = y$ ), we also have  $p^{-1}(w - 2a^{-1}wa) \in \mathbb{C}_y^\perp$ , hence  $x \perp y$  and thus  $xyx^{-1} = -y$ . Defining matrices  $\tilde{X}, \tilde{Y}$  using (4.20), these are of type (3.5) and perpendicular to  $V_g$  by (4.17) and (4.21).  $\square$

**Remark 2.** Clearly, the relations (4.6), (4.12), (4.14), (4.15) and (4.25) must be invariant under the action of  $U$ . In fact, if  $u = \left( \begin{pmatrix} q & \\ & 1 \end{pmatrix}, \begin{pmatrix} q & \\ & q \end{pmatrix} \right)$ , we have  $u.g = \tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$  with  $\tilde{a} = qaq^{-1}$  and  $\tilde{b} = qbq^{-1}$ .

Now we have proved the following

**Theorem 4.6.** *Let  $G = \text{Sp}(2)$  with the left invariant metric (2.7) and  $U \subset G \times G$  defined by (1.1). The orbit space  $M = G/U$  inherits a Riemannian metric such that the canonical projection  $\pi : G \rightarrow M$  is a Riemannian submersion. Let*

$$Z = \{p \in M; \exists \sigma \subset T_p M : \sec(\sigma) = 0\}.$$

*Then  $Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4$  where*

$$\begin{aligned} \pi^{-1}Z_1 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b \neq 0, \det(I - \text{Ad}(a^{-1}) - \text{Ad}(b^{-1})) = 0 \right\}, \\ \pi^{-1}Z_2 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; |a| = |b|, w := \text{Im } a^{-1}b \perp w - 2a^{-1}wa, |\text{Im } a| \geq |a|/2 \right\}, \\ \pi^{-1}Z_3 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; b = c = 0, |\text{Im } a| \geq 1/2 \right\}, \\ \pi^{-1}Z_4 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a = d = 0, |\text{Im } b| \geq 1/2 \right\}, \end{aligned}$$

*where all matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  are supposed to belong to  $\text{Sp}(2)$ .*  $\square$

**Remark 3.** The mistake in [4] is in the third line of the proof of the Theorem, page 1166. The computation of  $\langle v_g, X \rangle$  holds only for  $X \in \mathfrak{k}$ , but  $X$  may have a nonzero  $\mathfrak{p}$ -component as well. Thus the matrix  $X$  in (4), p. 1166, is too special and

must be replaced with the more general  $X = \begin{pmatrix} ry & -\bar{x} \\ x & -rxyx^{-1} \end{pmatrix}$  for arbitrary  $r \in \mathbb{R}$ , and instead of (5)  $\text{Im}(bx\bar{a}) = 0$  we obtain (5')  $\text{Im}(bx\bar{a}) = r(y - ay\bar{a})$ , while Equation (6)  $(ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1} = 0)$  remains unchanged. We have 15 variables,  $(a, b) \in S^7$ ,  $x \in \mathbb{H}$ ,  $y \in \text{Im}(\mathbb{H})$ ,  $r \in \mathbb{R}$ , with two arbitrary real constants (the lengths of  $x$  and  $y$ ), and 6 constraint equations (5') and (6) which reduce the number of free variables to 7. Thus the solution set is likely to project onto a subset with positive measure in the  $(a, b)$ -space  $S^7$ ; this would imply that the metric considered in [4] fails to have almost positive curvature.

## REFERENCES

- [1] A.L. Besse: *Einstein Manifolds*, Springer 1986
- [2] J. Cheeger: Some examples of manifolds of nonnegative curvature, *J. Diff. Geom.* **8** (1973), 223 - 268
- [3] J.-H. Eschenburg: *Freie isometrische Aktionen auf kompakten Liegruppen mit positiv gekrümmten Orbiträumen*, Schriftenreihe Math. Inst. Univ. Münster (2) 32 (1984)
- [4] J.-H. Eschenburg: Almost positive curvature on the Gromoll-Meyer 7-sphere, *Proc. Amer. Math. Soc.* **130** No. 4, 1165 - 1167
- [5] D. Gromoll, W.T. Meyer: An exotic sphere with nonnegative sectional curvature, *Ann. of Math.* **100** (1974), 401 - 406
- [6] K. Tapp: Flats in Riemannian submersions from Lie groups, *Preprint* (2007), DG0703389
- [7] F. Wilhelm: An exotic sphere with positive curvature almost everywhere, *J. Geom. Anal.* **11** (2001), 519 - 560
- [8] B. Wilking: Manifolds with positive sectional curvature almost everywhere, *Invent. Math.* **148** (2002), 117-141

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT AUGSBURG, D-86135 AUGSBURG, GERMANY  
*E-mail address*, (Eschenburg): [eschenburg@math.uni-augsburg.de](mailto:eschenburg@math.uni-augsburg.de)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 S 33RD ST., PHILADELPHIA, PA 19104, USA  
*E-mail address*, (M. Kerin): [mkerin@math.upenn.edu](mailto:mkerin@math.upenn.edu)